Functional Analysis

(alternative title: Math for Signals and Systems)

Course for MSc students at Tel Aviv University

Syllabus:

WEEK 1:
Vector spaces (axiomatic definition and properties, but assuming that matrix calculus is already known by the students). The emphasis is on infinite-dimensional spaces (signal spaces). We also do a quick revision of some concepts from analysis in n-dimensional spaces (distance, convergence, continuous functions, closed sets, bounded sets, compact sets).

WEEK 2:
More examples of vector spaces, subspaces, linearly independent sets, algebraic bases, examples of discrete-time and continuous-time signal spaces (l^p, convergent sequences, continuous functions on C^n and on intervals). We also do a quick revision of eigenvalues and eigenvectors of matrices.

WEEK 3:
Diagonalalizable matrices, their factorization (with detailed proofs). Analytic functions (assuming that the students have some background from years 1 and 2). Definition, different characterizations, examples and non-examples (sin, exp, logarithm), the concept of domain of analyticity, Taylor and Laurent series, convergence radius, Cauchy theorem, Morera's theorem, poles and isolated singularities, removable singularities, Cauchy's formula, maximum modulus theorem.

WEEK 4:
Normed spaces: definition, properties, many examples. Inner products, with many examples. Adjoint matrices, their diagonal factorization via unitary transformations. Positive and strictly positive matrices, how to test positivity.

WEEK 5:
The Cauchy-Schwarz inequality, the triangle inequality, when is a norm derived from an inner product? Complete normed spaces (Banach and Hilbert spaces), Cauchy sequences, the completion of an incomplete space, first introduction of L^p as the completion of continuous functions with the p-norm, Weierstrass theorems (the polynomials are dense in continuous functions, the trigonometric polynomials are dense in periodic continuous functions).

WEEK 6:
l^p spaces again, the completeness of l^1 (with detailed proof), introduction of L^1, sets of measure zero, equality almost everywhere.

WEEK 7:
The spaces L^p (p<infinity) and L^infinity, Holder inequality, various continuous inclusions, emphasis on L^2, examples.

WEEK 8:

The Hardy spaces H^2 and H^∞, on the unit disk, on the outside of the unit disk, on the right and left half-plane. The boundary function (defined almost everywhere on the unit circle or on the imaginary axis), equivalent expressions of the norm in terms of the boundary function, inner product on H^2, Bode plots.

WEEK 9:

Orthogonality in Hilbert spaces: the orthogonal complement and the closure of a set, convex sets, unique closest element in a closed convex set, Riesz projection theorem, Pythagoras identity, orthonormal bases, examples (mainly Fourier series in different versions), decomposing the time orthogonally into past and future.

WEEK 10:

Orthonormal bases in H^2 on the disk and half-plane. Linear operators, their space, their norm, isometric operators, unitary operators, projectors, shift operators on various spaces. Convolution operators, properties (norm, causality).

WEEK 11:

The Z-transform and the Laplace transform as unitary operators, i.e., the Paley-Wiener theorems (discrete-time version and continuous-time version). Time-invariant operators: definitions, examples, linear systems. The Foures-Segal theorem (discrete-time version and continuous-time version), i.e., the representation of time-invariant operators by transfer functions. Norms of transfer functions.

Sampling: its importance for information storage and transmission, the need to recover the original signal. Band-limited functions. The sampling theorem of Whittaker, Kotelnikov and Shannon (with proof).